A finite-certificate proof that nine points determine exactly twenty three-point unit circles

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Abstract

Let $a(n)$ be the maximum number of distinct unit circles containing at least three points of an n-point set in the Euclidean plane. This manuscript and its accompanying certificate bundle prove

$$\boxed{a(9) = 20}.$$

The upper bound is a finite exact elimination of all nine-point configurations with $m = 23, 22,$ or 21 counted circles. The lower bound is a new exact certificate for a nine-point configuration with exactly twenty counted unit circles. The construction is defined by a rational polynomial system obtained from the reflection equations of a $20T, s = 6$ all-triple template; a rational Krawczyk interval proves the existence and uniqueness of the required real root, and exact interval arithmetic excludes point collisions, center collisions, fourth points on intended circles, and all non-intended unit-radius triples. The bundle also retains the earlier exact $m = 19$ algebraic construction, the finite certificates used to eliminate $m = 23, 22, 21$, the scripts used to verify the $m = 20$ certificate, construction plots, and selected proven lower and upper bounds for larger n.

Contents

1 Problem statement

For a finite set $P \subset \mathbb{R}^2$, define

$$f(P) = \#\{\text{unit circles containing at least three points of } P\}.$$

The extremal function studied in OEIS A003829 is

$$a(n) = \max_{|P|=n} f(P).$$

A circle is counted once, regardless of whether it contains exactly three of the points or more than three of them. The previously known exact values used by the certificate search are

$$a(3), \ldots, a(8) = 1, 4, 4, 8, 12, 16,$$

as recorded in OEIS A003829 and the earlier unit-circle literature.[1, 3] The first open case was $n = 9$. The main result of this manuscript is that this case is now closed.

Theorem 1 (Main theorem)**.** *For nine points in the Euclidean plane,*

$$a(9) = 20.$$

The proof has two independent halves. The upper-bound half proves $a(9) \leq 20$ by eliminating $m = 23, 22, 21$. Here and throughout the upper-bound sections, $n = 9$ is fixed and m denotes the number of counted unit circles, not the number of points. The lower-bound half gives a certified configuration with exactly 20 counted unit circles.

1.1 Dual language

Given $P = \{p_0, \ldots, p_{n-1}\}$, draw the n unit circles

$$C_i = \{x \in \mathbb{R}^2 : \|x - p_i\| = 1\}.$$

A unit circle centered at q contains p_i if and only if $q \in C_i$. Thus counted unit circles in the original problem are exactly the 3-rich, 4-rich, etc. intersection points in the arrangement of the n dual unit circles. This dual language is useful because many constraints are linear in the point and center coordinates.

2 Pair slots and the first upper bound

For a counted circle C, let $s(C) = |C \cap P|$. The circle consumes $\binom{s(C)}{2}$ unordered point-pair slots. Each pair of points lies on at most two unit circles, so

$$\sum_C \binom{s(C)}{2} \leq 2\binom{n}{2}. \tag{1}$$

For $n = 9$ this gives a raw capacity of 72 pair slots.

One further slot is lost. Pinchasi's Gallai–Sylvester theorem for pairwise-intersecting unit circles[4] implies that, for $n \geq 5$, not all formal pair slots can be used by counted circles. Equivalently,

$$a(n) \leq \left\lfloor \frac{n(n-1)-1}{3} \right\rfloor \qquad (n \geq 5). \tag{2}$$

For $n = 9$ this gives

$$a(9) \leq \left\lfloor \frac{72-1}{3} \right\rfloor = 23.$$

The upper-bound certificates eliminate the remaining possibilities $m = 23, 22, 21$.

3 Finite certificate framework

This section summarizes the exact tests used in the finite upper-bound search.

3.1 Circle hypergraphs and profiles

A candidate with m counted circles is encoded by a hypergraph $\mathcal{H}$ on vertices $\{0, \ldots, 8\}$. A hyperedge is the support of a counted unit circle. We write T for a triple-circle, Q for a four-point circle, and R_5 for a five-point circle. Thus, for example, $19T + Q$ means nineteen triple-circles and one four-point circle.

For an all-triple profile, define the pair deficiency

$$\delta_{ij} = 2 - \operatorname{codeg}_{\mathcal{H}}(ij), \tag{3}$$

where $\operatorname{codeg}_{\mathcal{H}}(ij)$ is the number of triples containing the pair ij. Then $\delta_{ij} \geq 0$ and

$$\sum_{i<j} \delta_{ij} = 72 - 3m.$$

At every vertex,

$$\sum_{j \neq i} \delta_{ij} = 16 - 2\deg_{\mathcal{H}}(i),$$

so the weighted deficiency degree is even. Mixed profiles use the same pair-slot accounting, with a Q contributing one use to all six internal pairs and an R_5 contributing one use to all ten internal pairs.

3.2 Local caps

Every subset $S \subset \{0, \ldots, 8\}$ with $|S| \leq 8$ must induce no more counted circles than the known value $a(|S|)$. The exact local caps used are

$$a(3), \ldots, a(8) = 1, 4, 4, 8, 12, 16.$$

This inexpensive test removes many pair-feasible templates before metric information is used.

3.3 Four-point closure

Lemma 2 (Unit-circle closure on four points)**.** *Let A, B, C, D be four points. If three of the four triples among them lie on unit circles, then the fourth triple also lies on a unit circle.*

Proof. Translate A to the origin. Let x, y, z be the unit-circle centers for ABC, ABD, and ACD. Since B lies on the two unit circles through the origin centered at x and y, the two centers are reflections across the midpoint of $0B$, hence $B = x + y$. Similarly $C = x + z$ and $D = y + z$. Therefore B, C, D lie on the unit circle centered at $x + y + z$. □

When a four-point circle is present in a profile, all four of its three-subsets are supplied to the closure test.

3.4 Reflection equations

Suppose two distinct counted unit circles B and B' contain the same pair $\{i, j\}$. If their centers are $c_B, c_{B'}$, then the two possible unit-circle centers through p_i, p_j are reflections across the midpoint of $p_i p_j$. Therefore

$$c_B + c_{B'} = p_i + p_j. \tag{4}$$

These are homogeneous linear equations in one scalar coordinate at a time. Let $S(\mathcal{H})$ be the resulting scalar solution space. Constants always form a one-dimensional subspace. A non-collinear planar realization requires $\dim S(\mathcal{H}) \geq 3$ and cannot force $p_i = p_j$ for distinct original labels.

3.5 Affine Gram forms and rank two

After quotienting $S(\mathcal{H})$ by constants, a planar realization is equivalent to a positive semidefinite Gram form G of rank at most two on the quotient. For every intended incidence $p_i \in B$,

$$(c_B - p_i)^T G (c_B - p_i) = 1. \tag{5}$$

For fixed $\mathcal{H}$, these are linear equations in the entries of G. The first Gram test is therefore exact rational linear algebra. When an affine family of Gram forms remains, rank at most two is imposed by vanishing 3×3 minors. The terminal exact certificates either prove ideal infeasibility or show that every rank-two branch forces an original point collision.

3.6 Forced extra incidences

For an all-triple profile, if $v \notin B$ and

$$\|c_B - p_v\|^2 - 1 \equiv 0 \tag{6}$$

on every affine Gram solution, then any realization puts a fourth point on the intended triple-circle B. Such a template is not a genuine all-triple template and is eliminated from that profile.

4 Upper bound: elimination of $m = 23, 22, 21$

The following theorem is the upper-bound half of theorem 1.

Theorem 3. *No nine-point configuration has 21, 22, or 23 distinct unit circles containing at least three points. Hence $a(9) \leq 20$.*

4.1 The case $m = 23$

Under the 71-slot capacity, a four-point circle together with 22 further counted circles would use at least $6 + 22 \cdot 3 = 72$ slots. Thus an $m = 23$ candidate must be all triples. The total deficiency is $72 - 69 = 3$, and the even-degree condition forces the deficiency graph to be a triangle. Adding that missing triangle gives a twofold triple system on nine points.

The exact certificate checks all one-block deletions from the 36 non-isomorphic twofold triple systems on nine points. The terminal statuses are shown in table 1.

Table 1: Exact elimination of $m = 23$.

status	count
nonsimple remaining hypergraph	536
reflection nullity 1	163
reflection nullity 2	107
nullity 3 with forced point coincidence	50
nullity 4 with forced point coincidence	8
geometric survivors	0

Thus $m = 23$ is impossible.

4.2 The case $m = 22$

The only profiles compatible with the pair-slot bound are

$$22T, \qquad 21T + Q.$$

For $21T + Q$, fix $Q = \{0, 1, 2, 3\}$. The total deficiency is 3, and the weighted deficiency degree is odd exactly on the four vertices of Q. Up to the stabilizer of Q, there are two deficiency cases. The exact enumeration gives zero reflection survivors; see table 2 and `certificates/upper_bound/a003829_m22_elimination_summary.json`.

Table 2: Mixed $m = 22$ profile $21T + Q$.

case	pair-target	local cap	closure	nullity < 3	coinc.	survivors
0	33624	21480	2160	2112	48	0
1	5400	0	0	0	0	0

For $22T$, the total deficiency is 6. The local-cap and closure filters reduce the search to three missing-pair graph families: a six-cycle, two triangles sharing one vertex, and two disjoint triangles. The six-cycle family has the terminal exact counts in table 3. The two triangle-based classes are checked by adding the two missing triangle blocks and reducing to all two-block deletions from the 36 non-isomorphic twofold triple systems on nine points; those also have zero rank-two geometric survivors.

Therefore $m = 22$ is impossible.

Table 3: Six-cycle subcase of all-triple $m = 22$.

status	count
closure survivors	33930
reflection nullity < 3	31320
forced point coincidences	2412
Gram infeasible, nullity 3	72
Gram infeasible, nullity 4	90
rank-two branches forcing point coincidences	36
rank-two geometric survivors	0

4.3 The case $m = 21$

The possible profiles are

$$21T, \qquad 20T + Q, \qquad 19T + 2Q, \qquad 20T + R_5.$$

The mixed profiles are eliminated first, as summarized in table 4 and `certificates/upper_bound/a003829_m21_elimination_summary.json`.

Table 4: Mixed $m = 21$ profiles.

profile	exact obstruction
$20T + R_5$	With $R_5 = \{0, 1, 2, 3, 4\}$, the binary feasibility problem has 168 variables and 1164 constraints and is infeasible.
$19T + 2Q$	The genuine quad-intersection cases have intersection sizes 1 and 2. Ten weighted classes are enumerated. The only nonzero closure-survivor counts are $32, 348, 60, 84, 84$, and all are killed by reflection rank.
$20T + Q$	With $Q = \{0, 1, 2, 3\}$, 15541 labelled fixed-Q patterns and 53 weighted classes give 381559 completions, 214886 local-cap survivors, 32116 closure survivors, 116 reflection survivors, and zero rank-two Gram survivors.

It remains to eliminate $21T$. The all-triple total deficiency is 9, and the weighted deficiency degree is even at every vertex. The initial enumeration has 74 weighted deficiency classes, 1698371 pair-target completions, 1388056 local-cap survivors, 394206 closure survivors, and 49057 reflection survivors. The first Gram pass eliminates 33544 nullity-3 cases and 5297 nullity-4 cases as Gram-infeasible; 4073 nullity-4 cases have no rank-two solution; 3808 PSD-degenerate nullity-3 cases are rank-one line configurations and hence impossible for three distinct points on a unit circle. A higher-nullity residual of 2335 cases is then split by the number s of single-deficient pairs.

The final deficiency-family refinements are in table 5. The last seven residual template orbits, three from $s = 7$ and four from $s = 9$, are all killed by exact forced extra-incidence identities in `a003829_round14_extra_incidence_certificate.json`.

Combining the mixed-profile and all-triple eliminations proves $m = 21$ impossible. Since $m = 23$ and $m = 22$ were already eliminated, theorem 3 follows.

5 The earlier exact lower bound with nineteen circles

Before the twenty-circle construction was found, the best certified lower bound was an exact algebraic configuration with nineteen counted circles. It remains in this bundle because it explains part of the search process and gives a useful independent check on the algebraic machinery.

Let α be the unique real root in $(-0.926, -0.925)$ of

$$\begin{aligned} P(t) =& 16360000t^{12} + 46507200t^{11} + 77796336t^{10} + 65943904t^9 + 44741196t^8 \\ &+ 35333820t^7 + 323349485t^6 + 19478148t^5 + 2552448t^4 + 4540352t^3 \\ &+ 1463520t^2 - 324480t - 86528. \end{aligned}$$

Let

$$\begin{aligned} N(t) =& 16713476207714287600000t^{11} + 44669020988871784352000t^{10} \\ &+ 721323179029483834017 6t^9 + 56206270902990585369 12t^8 \\ &+ 38387744655076676170 12t^7 + 32378666645610473560 96t^6 \\ &+ 29798852580907438779 43t^5 + 16613117020410550399 57t^4 \\ &+ 11015495900029532335 4t^3 + 54455159801142690082 4t^2 \\ &+ 10229434574283807667 2t - 26378267444417965024, \end{aligned}$$

and set $\beta = N(\alpha)/9361987027029000000$. With

$$\phi(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right),$$

define

$$\begin{gathered} X_0 = (1,0), \quad X_3 = (-1,0), \quad X_1 = \phi(\alpha), \quad X_2 = \phi(\beta), \\ U = (-3/5, -4/5), \qquad V = (-5/13, -12/13), \\ X_4 = X_2 + (1,0) + U, \qquad X_5 = X_2 + (1,0) + V. \end{gathered}$$

The nine points are the rows of AX, where X is the 6×2 matrix with rows $X_0, \ldots, X_5$ and

$$A = \begin{pmatrix} -\frac{1}{2} & 1 & 0 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & -1 & \frac{1}{2} & 1 & 1 \\ -1 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 1 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 1 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

The nineteen triples are

$$\begin{gathered} 012,\ 028,\ 045,\ 047,\ 058, \quad 125,\ 135,\ 234,\ 238,\ 246, \\ 256,\ 345,\ 378,\ 468,\ 478, \quad 568,\ 017,\ 137,\ 036. \end{gathered}$$

The standard-library verifier `scripts/verify_m19_n10_exact.py` works in $\mathbb{Q}(\alpha)$, evaluates the point-only circumradius polynomial for all $\binom{9}{3}$ triples, proves that the zero set is exactly the nineteen triples above, and interval-certifies non-collinearity and distinct centers. It also verifies the earlier ten-point extension with exactly twenty-two three-point unit circles.

Table 5: Final all-triple $m = 21$ refinements.

case	certificate summary	survivors
$s = 3$	24 weighted classes give 144212 pair-target completions, 19424 local-cap/closure survivors, 48 reflection survivors; all 48 force point collisions in the Gram stage.	0
$s = 5$	15 weighted classes give 31212 local-cap/closure survivors, 350 reflection survivors, 16 Gram-linear residuals; exact rank-two minors kill all residuals.	0
$s = 7$	22 weighted classes give 133038 local-cap/closure completions, 6610 reflection survivors, 30 Gram-linear residuals, then 3 residual orbits; all 3 force extra incidences.	0
$s = 9$	The deficiency graph is a simple Eulerian 9-edge graph. After closing the 9-cycle, triangle-decomposable classes, and further Gram eliminations, 4 residual orbits remain; all 4 force extra incidences.	0

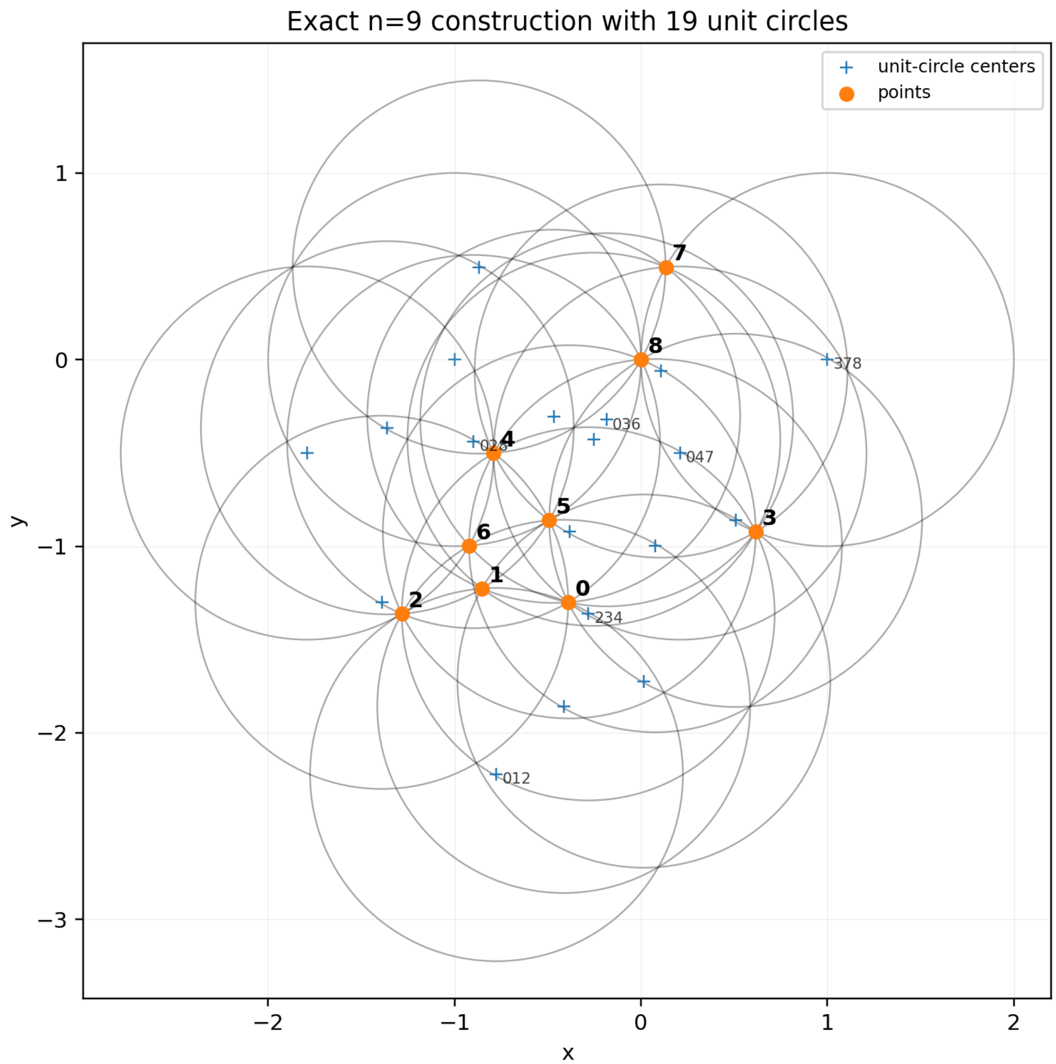

Figure 1: The earlier exact $n = 9$, $m = 19$ construction. Points are labelled; plus signs mark unit-circle centers.

6 The certified twenty-circle construction

This section gives the lower-bound half of theorem 1. The construction was found in the all-triple $20T, s = 6$ deficiency layer. In that layer, the search considered 266 weighted deficiency classes. After closure, reflection, nullity, affine-Gram, and forced-extra-incidence tests, the terminal pipeline had 46 affine-Gram residual cases, concentrated in classes $9, 10, 87, 88, 140, 239, 262$; class 140 yielded the isolated rank-two realization certified here. These search statistics are not part of the existence proof, but they record how the final certificate was reached.

6.1 Incidence pattern

The twenty intended triples are

$$\begin{gathered} 012,\ 037,\ 038,\ 046,\ 047,\ 056,\ 058, \\ 136,\ 137,\ 146,\ 148,\ 157,\ 158, \\ 236,\ 238,\ 247,\ 248,\ 256,\ 257,\ 345. \end{gathered} \tag{7}$$

Partition the points as

$$A = \{0, 1, 2\}, \qquad B = \{3, 4, 5\}, \qquad C = \{6, 7, 8\}.$$

The pattern consists of the two intra-group triples 012 and 345, together with all cross triples $(a, b, c) \in A \times B \times C$ except the nine with $a + b + c \equiv 0 \pmod 3$ after indexing each group by $0, 1, 2$. Its deficiency pattern has single deficiencies on $01, 02, 12, 34, 35, 45$ and double deficiencies on $67, 68, 78$, so it belongs to the all-triple $20T, s = 6$ layer.

For orientation only, the isolated real root has the approximate coordinates in table 6. The exact construction is not defined by these decimals; it is defined by the rational polynomial certificate below.

Table 6: Approximate coordinates of the certified $m = 20$ construction.

i	x_i	y_i
0	0.0000000000000000	0.0000000000000000
1	1.0978941215225699	0.0000000000000000
2	-0.4505445972749385	-0.8677385486715592
3	0.2071734593478844	0.6025968098608510
4	0.4412929999324647	-0.4821327244762682
5	-0.0011169350327177	-0.9882026340561421
6	0.3923352634888004	-1.3742531016840637
7	0.1250000000000000	1.0065145530125045
8	0.1300142607588310	-0.5000000000000000

6.2 Exact polynomial system

Let

$$\xi = (X_0, \ldots, X_9, Y_0, \ldots, Y_9) \in \mathbb{R}^{20}.$$

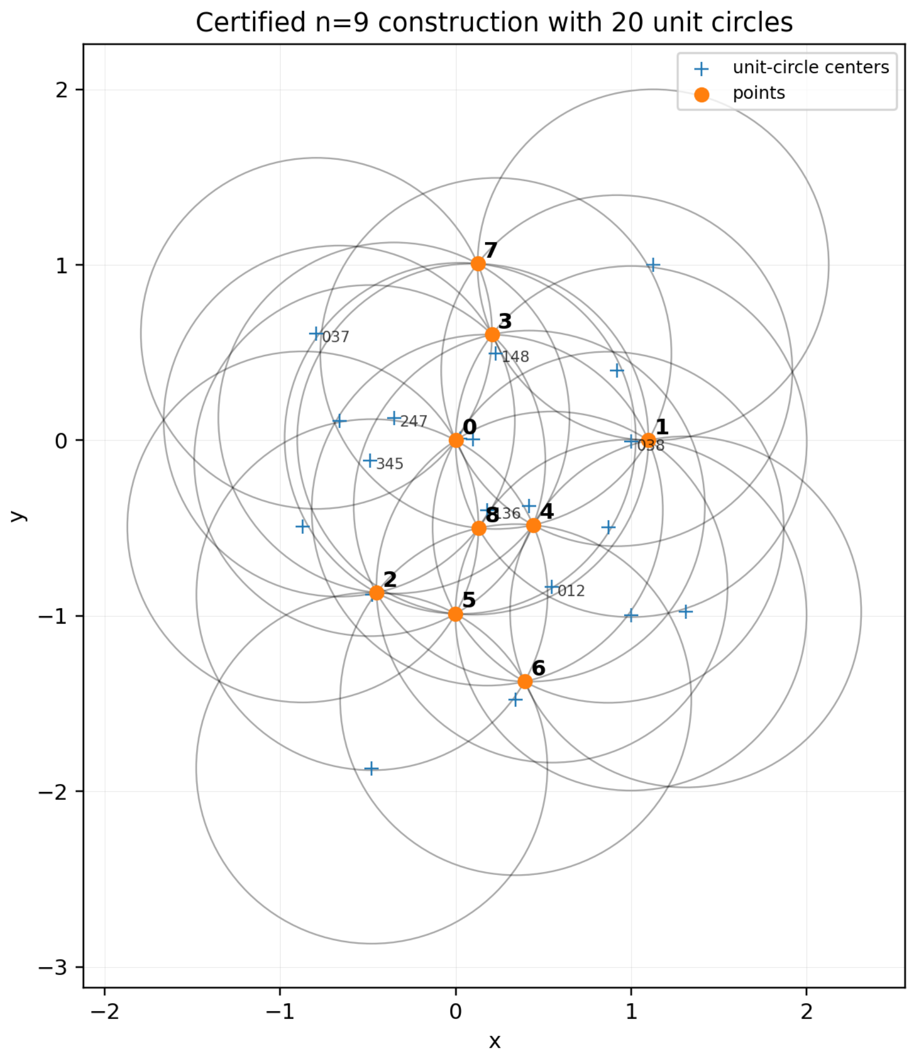

Figure 2: The certified $n = 9$, $m = 20$ construction. Points are labelled; plus signs mark the twenty distinct unit-circle centers.

The certificate file `certificates/m20_s6_formal_certificate.json` stores a rational 29×10 matrix N. Rows $0, \dots, 8$ define the original points and rows $9, \dots, 28$ define the twenty intended centers in the order of equation (7). Namely,

$$Z_r(\xi) = \left(\sum_{k=0}^{9} N_{rk} X_k,\ \sum_{k=0}^{9} N_{rk} Y_k \right), \qquad r = 0, \dots, 28.$$

Then $P_i = Z_i$ for $0 \le i \le 8$, and the center of the jth listed triple is $Q_j = Z_{9+j}$.

The reflection equations reduce all sixty intended point-center incidences to fifteen displacement rows $r_1, \dots, r_{15} \in \mathbb{Q}^{10}$, up to sign. The rows are:

$$\begin{aligned}
r_1 &= (1, -\tfrac{3}{2}, 1, -1, -1, \tfrac{3}{2}, -1, \tfrac{1}{2}, \tfrac{1}{2}, 0),\\
r_2 &= (1, -\tfrac{1}{2}, -1, 0, 0, -\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0),\\
r_3 &= (1, \tfrac{1}{2}, 0, 0, 0, -\tfrac{1}{2}, 0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0),\\
r_4 &= (0, \tfrac{1}{2}, 0, 0, 0, -\tfrac{1}{2}, 1, -\tfrac{1}{2}, -\tfrac{1}{2}, 0),\\
r_5 &= (0, \tfrac{1}{2}, -1, 0, 0, -\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0),\\
r_6 &= (0, \tfrac{1}{2}, -1, 1, 1, -\tfrac{3}{2}, 0, \tfrac{1}{2}, -\tfrac{1}{2}, 0),\\
r_7 &= (0, \tfrac{1}{2}, 0, 1, 0, -\tfrac{1}{2}, 0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0),\\
r_8 &= (0, \tfrac{1}{2}, 0, 0, 0, -\tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0),\\
r_9 &= (0, \tfrac{1}{2}, -1, 0, 1, -\tfrac{1}{2}, 1, -\tfrac{1}{2}, -\tfrac{1}{2}, 0),\\
r_{10} &= (0, \tfrac{1}{2}, 0, 0, 0, -\tfrac{1}{2}, 0, \tfrac{1}{2}, -\tfrac{1}{2}, 0),\\
r_{11} &= (0, \tfrac{1}{2}, 0, 0, 1, -\tfrac{1}{2}, 0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0),\\
r_{12} &= (0, \tfrac{1}{2}, 0, 0, 0, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0),\\
r_{13} &= (0, \tfrac{1}{2}, 0, 1, 1, -\tfrac{1}{2}, 0, -\tfrac{1}{2}, -\tfrac{1}{2}, -1),\\
r_{14} &= (0, \tfrac{1}{2}, 0, 0, 0, \tfrac{1}{2}, 1, -\tfrac{1}{2}, -\tfrac{1}{2}, -1),\\
r_{15} &= (0, \tfrac{1}{2}, 0, 0, 0, -\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, -1).
\end{aligned}$$

The fifteen unit equations are

$$(r_\nu \cdot X)^2 + (r_\nu \cdot Y)^2 = 1, \qquad \nu = 1, \dots, 15. \tag{8}$$

The five gauge equations are

$$Z_0^x = 0, \qquad Z_0^y = 0, \qquad Z_1^y = 0, \qquad Z_7^x = \frac{1}{8}, \qquad Z_8^y = -\frac{1}{2}.$$

Together these form twenty rational polynomial equations in twenty real variables.

6.3 Root isolation by a rational Krawczyk certificate

Let $F : \mathbb{R}^{20} \to \mathbb{R}^{20}$ be the polynomial map whose components are the fifteen equations (8) and the five gauge equations. The certificate stores a rational center $c \in \mathbb{Q}^{20}$, the box

$$X = c + [-2^{-180}, 2^{-180}]^{20},$$

and a rational 20×20 matrix C. The verifier evaluates the Krawczyk operator

$$K(c, X) = c - CF(c) + (I - CJ(X))(X - c)$$

with exact rational interval arithmetic and proves

$$K(c, X) \subset \operatorname{int}(X).$$

It also proves

$$\|I - CJ(X)\|_\infty < 2.13 \cdot 10^{-52} < 1.$$

Therefore the system has a unique real zero ξ^* in X.

6.4 Incidences, separations, and exclusion of extras

For each intended incidence $Q_j P_i$, the verifier computes the exact row of N defining $Q_j - P_i$ and checks that it is equal, up to sign, to one of the fifteen rows r_ν. Hence every intended incidence has distance exactly one at the isolated root.

The same rational interval calculation proves the following margins throughout the root box:

$$\begin{aligned} \min_{i<j} \|P_i - P_j\|^2 &> 0.09721369299616801, \\ \min_{a<b} \|Q_a - Q_b\|^2 &> 0.005197862310899273\!5, \\ \min_{i \notin T_j} \left| \|Q_j - P_i\|^2 - 1 \right| &> 0.06577095439165559. \end{aligned}$$

Thus the nine points are distinct, the twenty intended centers are distinct, and no intended circle contains a fourth point.

For every non-listed triple i, j, k, the verifier interval-evaluates the exact point-only unit-circumradius polynomial

$$\begin{aligned} \Phi_{ijk} = \|P_j - P_k\|^2 \|P_i - P_k\|^2 \|P_i - P_j\|^2 \\ - 4\,\mathrm{cross}(P_i, P_j, P_k)^2. \end{aligned} \tag{9}$$

For a non-collinear triple, $\Phi_{ijk} = 0$ is equivalent to circumradius one. All 64 non-listed triples have intervals excluding zero. The weakest certified bound is

$$|\Phi_{458}| > 0.07832522266666986.$$

Theorem 4. *There exists a set of nine points in the Euclidean plane with exactly twenty distinct unit circles containing at least three of the points. Hence $a(9) \geq 20$.*

Proof. The isolated zero ξ^* defines the points $P_0, \ldots, P_8$ and centers $Q_0, \ldots, Q_{19}$ by the rational linear map N. The unit-row check proves that every listed triple in (7) lies on its listed unit circle. The center-separation bound proves these twenty circles are distinct. The nonincident center-point bound proves none of the twenty intended circles contains a fourth point. Finally, (9) excludes every non-listed unit-radius triple. Therefore the configuration has exactly twenty counted unit circles. □

Proof of theorem 1. Theorem 3 proves $a(9) \leq 20$, and theorem 4 proves $a(9) \geq 20$. □

7 Completion extensions and larger-n bounds

The $m = 20$ construction also gives improved exact lower bounds for $n = 10$ and $n = 11$.

Lemma 5 (Four-point completion from a unit triple)**.** *Let A, B, C lie on a unit circle centered at Q. Put*

$$D = A + B + C - 2Q.$$

Then ABD, ACD, and BCD also lie on unit circles. Their centers are

$$A + B - Q, \qquad A + C - Q, \qquad B + C - Q.$$

Proof. Translate A to the origin. With $x = Q - A$, write $B = x + y$ and $C = x + z$ where $\|x\| = \|y\| = \|z\| = 1$. Then $D = y + z$ in translated coordinates. The triples ABD, ACD, and BCD have centers y, z, and $x + y + z$, respectively. □

Completing the 012 circle in the certified $m = 20$ construction gives three additional distinct counted circles, so

$$a(10) \geq 23.$$

Completing both the 012 and 345 circles gives six additional distinct counted circles, so

$$a(11) \geq 26.$$

The script `scripts/verify_m20_completion_extensions.py` checks the exact unit-row identities and interval-certifies that the resulting $20+6$ centers are distinct. Adding arbitrary further generic points gives $a(n) \geq 26$ for every $n \geq 11$; stronger selected lattice lower bounds are listed below.

7.1 Selected proven bounds

For $n \geq 10$ we use the general upper bound (2). The lower bounds in table 7 are exact: the first rows come from known exact values and the certified $m = 20$ construction; the later rows come from square- or triangular-lattice point sets with exact rational radius certificates. The lattice verifier counts exact circumcenters at the recorded squared radius, and scaling by the reciprocal of that radius gives unit circles.

8 Reproducibility and bundle contents

The bundle is designed to be checked offline. From the extracted bundle root, run

```
python3 scripts/run_all_checks.py # alias: python3 scripts/verify_all.py
```

This executes the following proof checks:

1. `scripts/replay_upper_bound_summaries.py`: checks the terminal $m = 23, 22, 21$ certificate summaries used for $a(9) \leq 20$.
2. `scripts/verify_m20_s6_formal_certificate.py`: verifies the rational Krawczyk certificate and the exact "exactly twenty" geometry check.
3. `scripts/verify_m20_completion_extensions.py`: verifies $a(10) \geq 23$ and $a(11) \geq 26$ from four-point completions of the $m = 20$ construction.
4. `scripts/verify_m19_n10_exact.py`: verifies the older $m = 19$ construction and its $n = 10$ extension.
5. `scripts/verify_lattice_lower_bounds.py`: verifies selected square- and triangular-lattice lower-bound rows exactly.

The proof-critical scripts use only Python's standard library. Figure generation uses Matplotlib but is not part of the proof.

The main directories are:

- `manuscript/`: this TeX source, the compiled PDF, and figure files.
- `certificates/`: the exact $m = 20$ certificate and the upper-bound certificate summaries.
- `data/`: construction data and selected larger-n bound data.

Table 7: Selected proven bounds bundled with the manuscript. For $n \geq 10$, the upper bound is the integer part of $(n(n-1)-1)/3$.

n	lower	upper	gap	lower-bound source
3	1	1	0	known exact
4	4	4	0	known exact
5	4	4	0	known exact
6	8	8	0	known exact
7	12	12	0	known exact
8	16	16	0	known exact
9	20	20	0	this manuscript
10	23	29	6	one $m = 20$ completion
11	26	36	10	two $m = 20$ completions
12	26	43	17	monotonicity
13	29	51	22	exact triangular lattice
14	32	60	28	exact triangular lattice
15	37	69	32	exact triangular lattice
16	40	79	39	exact triangular lattice
17	44	90	46	exact triangular lattice
18	47	101	54	exact triangular lattice
19	51	113	62	exact triangular lattice
20	54	126	72	exact triangular lattice
21	58	139	81	exact square lattice
22	62	153	91	exact square lattice
23	65	168	103	exact square lattice
24	69	183	114	exact square lattice
25	74	199	125	exact square lattice
30	100	289	189	exact square lattice
40	164	519	355	exact triangular lattice
50	233	816	583	exact triangular lattice
61	327	1219	892	exact triangular lattice
80	462	2106	1644	exact triangular lattice
100	611	3299	2688	exact triangular lattice
120	715	4759	4044	exact triangular lattice
140	959	6486	5527	exact triangular lattice
160	1238	8479	7241	exact triangular lattice
169	1350	9463	8113	exact triangular lattice

- `scripts/`: standalone checkers and the top-level replay script.
- `legacy_scripts/`: scripts retained from the pre-construction upper-bound archive.
- `raw_inputs/`: the pre-construction round-21 archive, retained for provenance.

References

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[2] P. Erdős and G. Purdy, *Extremal problems in combinatorial geometry*, Chapter 17 in *Handbook of Combinatorics*, Elsevier, 1995.

[3] H. Harborth and I. Mengersen, *Point sets with many unit circles*, Discrete Mathematics 60 (1986), 193–197, doi:10.1016/0012-365X(86)90011-7.

[4] R. Pinchasi, *Gallai–Sylvester theorem for pairwise-intersecting unit circles*, Discrete & Computational Geometry 28 (2002), 607–624, doi:10.1007/s00454-002-2892-3.

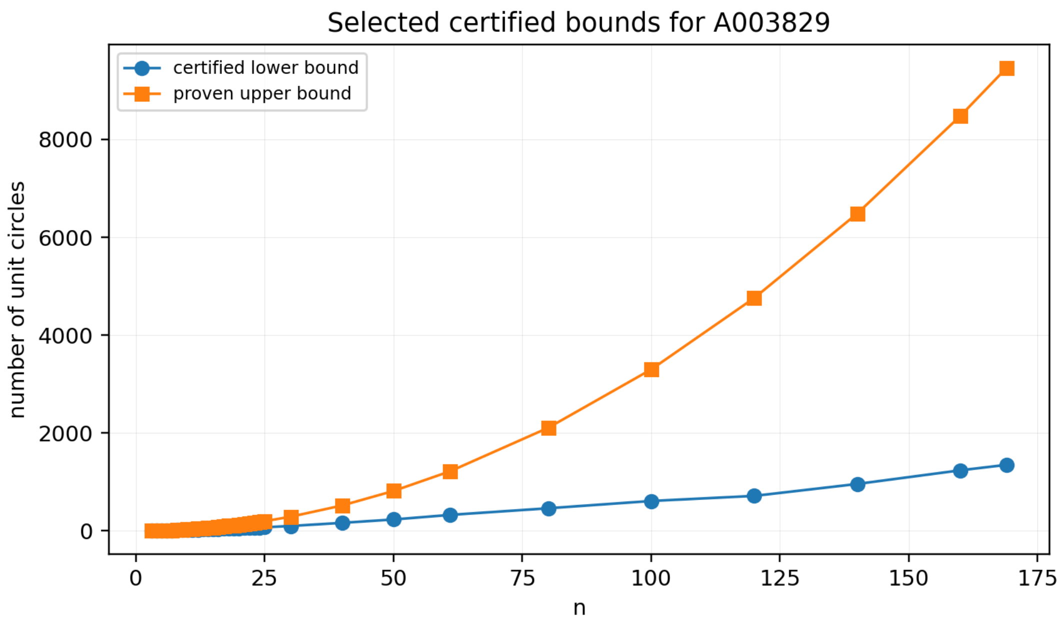

Figure 3: Selected proven lower and upper bounds from table 7.