

THE EXACT REGION FORMULA FOR ARRANGEMENTS OF CONSTRAINED LONG-LEGGED A'S

SIDDHARTHA MAHAJAN AND PARAS CHOPRA

ABSTRACT. A constrained long-legged A consists of two rays with a common apex and a crossbar joining points at equal distance from that apex. Let $a(n)$ be the maximum number of regions into which n such objects divide the plane. Cutler, Karlsson, and Sloane established that two constrained A's meet at most eight times and that three of them cannot be pairwise eight-crossing. The graph of eight-crossing pairs is therefore triangle-free, and Mantel's theorem gives

$$a(n) \leq 7 \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor + 2n + 1.$$

We construct matching arrangements for every n . The construction splits the objects into a narrow family and a wide nested fan. Every cross-family pair has eight proper intersections, while every same-family pair has at least seven. Three elementary blow-ups, together with one cubic Taylor calculation for the narrow family, verify all required component incidences. Because these inequalities are strict, a generic rational perturbation preserves them. Consequently, for every $n \geq 0$,

$$a(n) = 7 \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor + 2n + 1 = \left\lfloor \frac{15n^2 - 6n + 4}{4} \right\rfloor.$$

1. INTRODUCTION

A constrained long-legged A is the union of two rays issuing from one apex and a crossbar whose endpoints lie at equal distance from the apex. The region-maximization problem for these objects was introduced by Cutler, Karlsson, and Sloane in their study of planar cuts by exotic knives [1]; the resulting sequence is OEIS A397182 [4]. The equality of the two leg lengths is a metric condition, so affine deformations that trivialize many line-arrangement problems are unavailable here.

Let $a(n)$ denote the largest number of connected components of the complement of a union of n constrained A's. The first values found in [1] are

$$a(0), a(1), a(2), a(3) = 1, 3, 13, 30.$$

The main result of this paper determines the sequence completely.

Theorem 1.1. *For every integer $n \geq 0$,*

$$(1) \quad a(n) = 7 \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor + 2n + 1 = \left\lfloor \frac{15n^2 - 6n + 4}{4} \right\rfloor.$$

The upper bound is an extremal-graph argument. Join two objects when they meet eight times. The three-object result of [1] makes this graph triangle-free, so Mantel's theorem bounds the number of eight-pairs. Equality forces a balanced complete bipartite graph of eight-pairs and seven crossings on every remaining pair.

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The lower construction realizes exactly that colored graph. For prescribed part sizes p, q , we build p narrow A's and q wide A's. The geometry is organized around two points, $O = (0, 0)$ and $C = (1, 0)$, and is read on three scales. Near O , the narrow legs form a regular fan. Near C , the narrow crossbars become a short vertical stack and the high legs and crossbars of the wide family become a nested fan. Near the wide apices, a second blow-up orders the low legs. These limits establish seven incidences on every same-family pair and eight on every cross-family pair. The only limiting incidences that land at component endpoints are two narrow bar-leg hits; a third-order expansion places both strictly inside the bars.

The proof is entirely analytic. Exact rational certificates through $n = 16$, which arose during discovery, are included as supplementary checks but are not used in the proof of theorem 1.1. Colored extremal graphs also play a central role in the related published lollipop-bounds paper [2].

2. THE GEOMETRIC MODEL AND THE UPPER BOUND

2.1. Constrained A's and proper crossings. For an angle ϕ , write

$$\mathbf{e}(\phi) = (\cos \phi, \sin \phi).$$

Given an apex $P \in \mathbb{R}^2$, a radius $r > 0$, and two distinct unit vectors u, v , define

$$(2) \quad A(P, r; u, v) = \{P + tu : t \geq 0\} \cup \{P + tv : t \geq 0\} \cup [P + ru, P + rv].$$

The first two components are the legs and the last is the crossbar. We always label the legs so that $\det(u, v) > 0$; their counterclockwise angular separation then lies in $(0, \pi)$.

An intersection of components belonging to different A's is *proper* when it lies strictly past the apex on every participating leg and strictly inside every participating crossbar. An arrangement is *generic* if supporting lines of components from different A's are nonparallel, every actual cross-shape component intersection is proper, no vertex lies on a foreign component, and no point lies on components from three or more A's. It is enough to maximize over generic arrangements: transverse proper crossings persist under perturbation, while a local generic resolution of a parallelism, vertex incidence, or multiple crossing weakly increases the number of complementary regions. We will construct an open set with the desired pairwise incidences and then choose a generic point in that set.

2.2. The region identity.

Proposition 2.1. *A generic arrangement of n constrained A's with I finite crossings has*

$$F = I + 2n + 1$$

complementary regions.

Proof. Compactify the plane by one point at infinity. One constrained A becomes a connected graph with four vertices (the apex, the two crossbar endpoints, and infinity) and five edges, hence first Betti number two. The union of n disjoint A's with their points at infinity identified has first Betti number $2n$. Each proper transverse crossing identifies two interior points of distinct arcs and increases the first Betti number by one. Thus the compactified union has first Betti number $2n + I$. Euler's formula on the sphere gives $F = (2n + I) + 1$. $\square$

2.3. The eight-crossing graph. We use the following two geometric facts established in [1, Sec. 13.2].

Proposition 2.2 (Cutler–Karlsson–Sloane). *Two constrained long-legged A's have at most eight proper finite intersections. Moreover, no three constrained A's are pairwise eight-crossing.*

For a generic arrangement, let G_8 be the graph whose vertices are the A's and whose edges are the pairs with eight crossings. By proposition 2.2, G_8 is triangle-free.

Theorem 2.3 (Universal upper bound). *For every $n \geq 0$,*

$$(3) \quad a(n) \leq 7 \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor + 2n + 1.$$

If equality holds, the eight-crossing graph is a balanced complete bipartite graph and every nonedge corresponds to a seven-crossing pair.

Proof. Let $e = e(G_8)$. Every edge contributes at most eight crossings and every nonedge at most seven, so

$$I \leq 8e + 7 \left(\binom{n}{2} - e \right) = 7 \binom{n}{2} + e.$$

Mantel's theorem [3] gives $e \leq \lfloor n^2/4 \rfloor$. Combining this inequality with proposition 2.1 proves (3). Equality requires equality in every step. The equality case of Mantel's theorem is the balanced complete bipartite graph, and every nonedge must contribute seven crossings. $\square$

It remains to realize the equality pattern for arbitrary part sizes.

3. A STABILITY PRINCIPLE FOR COMPONENT INCIDENCES

We will repeatedly inspect a family of components after an affine rescaling that depends on a small parameter $\varepsilon > 0$. The following elementary observation formalizes the limiting arguments.

Lemma 3.1 (Blow-up stability). *Let two supporting lines and their directed component domains depend continuously on $\varepsilon > 0$. Suppose that after an invertible affine change of coordinates they converge, as $\varepsilon \rightarrow 0^+$, to two nonparallel lines whose intersection lies in the relative interior of both limiting domains. Then, for all sufficiently small $\varepsilon > 0$, the original components have a proper intersection.*

Proof. The intersection parameters of two nonparallel lines are continuous rational functions of their origins and directions. At the limiting configuration both parameters satisfy strict domain inequalities. Those inequalities therefore persist for all sufficiently small ε . $\square$

All asymptotics below are taken with the finite integers p, q fixed. Thus an $O(\varepsilon^r)$ estimate is uniform over every index occurring in the construction, and finitely many smallness requirements can be met by one choice of ε .

For two A's, order the components as (first leg, second leg, crossbar) and write a 3×3 0–1 matrix for their proper component incidences. Our construction will establish the following entrywise lower bounds:

$$(4) \quad M_{NN} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_{NW} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_{WW} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

They contain respectively seven, eight, and seven required incidences. A zero in (4) means only that no incidence is required there.

4. THE TWO-FAMILY CONSTRUCTION

Fix integers $p, q \geq 1$ and put $m = \max\{p, q\}$. Let

$$O = (0, 0), \quad C = (1, 0), \quad h = \frac{3}{2}.$$

We define p narrow A's $N_1, \dots, N_p$ and q wide A's $W_1, \dots, W_q$, depending on a sufficiently small parameter $\varepsilon > 0$.

4.1. The narrow family. For $1 \leq i \leq p$, set

$$a_i = 3i, \quad \alpha_i = a_i \varepsilon, \quad r_i = \cos \alpha_i, \quad P_i = C - r_i \mathbf{e}(\alpha_i).$$

The two leg directions of N_i are

$$d_i^- = \mathbf{e}(\alpha_i - \varepsilon), \quad d_i^+ = \mathbf{e}(\alpha_i + \varepsilon),$$

and the crossbar joins $P_i + r_i d_i^-$ to $P_i + r_i d_i^+$. For small ε , $r_i > 0$ and this is a constrained A.

It is useful to write its crossbar in centered form. Define

$$n_i = \mathbf{e}(\alpha_i + \frac{\pi}{2}), \quad M_i = C - r_i(1 - \cos \varepsilon) \mathbf{e}(\alpha_i), \quad s_i = r_i \sin \varepsilon.$$

Then the crossbar is

$$(5) \quad \mathcal{B}_i = \{M_i + t n_i : -s_i \leq t \leq s_i\}.$$

Its supporting line has the particularly simple equation

$$(6) \quad \mathbf{e}(\alpha_i) \cdot X = \cos \alpha_i \cos \varepsilon.$$

4.2. The wide family. For $1 \leq j \leq q$, put

$$(7) \quad q_j = 4^{-j}, \quad \theta_j = 2 \arctan q_j, \quad D_j = q_j^{-1} = 4^j, \quad L_j = 4m + j, \quad \lambda_j = L_j \varepsilon.$$

The low and high leg directions are

$$(8) \quad \ell_j = \mathbf{e}(\lambda_j), \quad u_j = \mathbf{e}(\pi - \theta_j + \lambda_j).$$

Place the high crossbar endpoint at

$$H_j = C + \varepsilon(-D_j, h)$$

and define

$$(9) \quad R_j = \frac{1 + h\varepsilon}{\sin(\theta_j - \lambda_j)}, \quad Q_j = H_j - R_j u_j.$$

Then $(Q_j)_y = -1$. The wide A W_j has apex Q_j , common leg length R_j , low leg direction ℓ_j , and high leg direction u_j . Its low crossbar endpoint is

$$K_j = Q_j + R_j \ell_j.$$

For small ε , $0 < \lambda_j < \theta_j$, so $R_j > 0$ and the two directions in (8) are distinct with positive orientation.

At $\varepsilon = 0$ the relevant apex and endpoint abscissae are

$$(10) \quad A_j = 1 + \cot \theta_j = 1 + \frac{D_j - q_j}{2}, \quad B_j = 1 + \cot \frac{\theta_j}{2} = 1 + D_j.$$

Indeed,

$$(11) \quad Q_j \longrightarrow (A_j, -1), \quad K_j \longrightarrow (B_j, -1), \quad H_j \longrightarrow C.$$

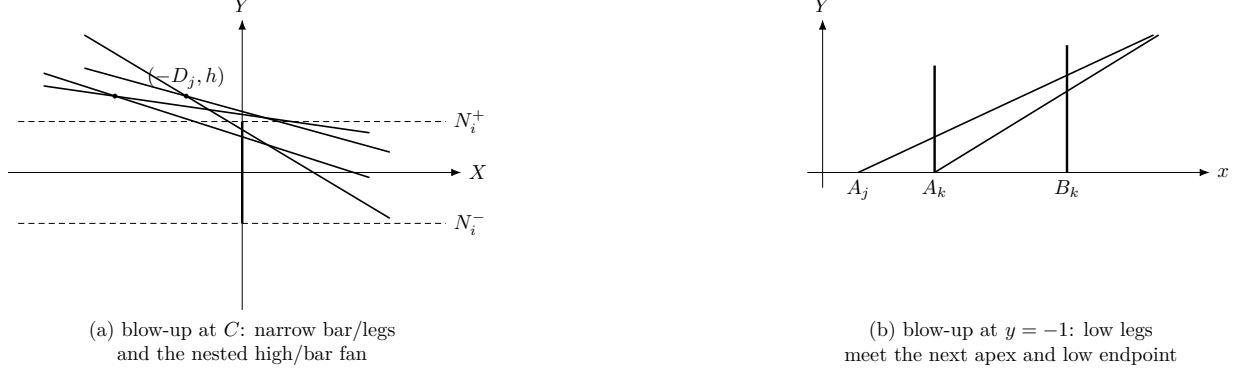


FIGURE 1. The two limiting charts used for narrow–wide and wide–wide incidences. The drawing is schematic; only the strict order and slope relations are used.

5. VERIFICATION OF THE THREE PAIR TYPES

5.1. Narrow–narrow pairs.

Lemma 5.1. *For $1 \leq i < k \leq p$ and all sufficiently small $\varepsilon > 0$, the component incidence matrix of (N_i, N_k) contains M_{NN} from (4). In particular, the pair has at least seven proper intersections.*

Proof. Write $a = a_i$, $b = a_k$, so $b - a \geq 3$. First apply the blow-up $(x, y) \mapsto (x, y/\varepsilon)$ near O . The leg of N_i with sign $\sigma \in \{-1, +1\}$ converges to the directed line

$$(12) \quad Y = -a + (a + \sigma)x, \quad x > 0.$$

For signs σ, τ , the limiting intersection of the corresponding legs of N_i, N_k has

$$x = \frac{b - a}{b - a + \tau - \sigma} > 0,$$

because the denominator is at least $(b - a) - 2 \geq 1$. By lemma 3.1, all four leg–leg intersections are proper.

Next consider the intersection of $\mathcal{B}_i$ with the τ -leg of N_k . Put $A = a\varepsilon$, $B = b\varepsilon$ and write the point on the bar as $M_i + tn_i$. Direct line intersection gives a ray parameter

$$(13) \quad \rho_\tau = \frac{\cos A(\cos \varepsilon - 1) + \cos B \cos(B - A)}{\cos(B - A + \tau\varepsilon)} = 1 + O(\varepsilon^2)$$

and bar coordinate

$$(14) \quad t_\tau = -\cos B \sin(B - A) + \rho_\tau \sin(B - A + \tau\varepsilon).$$

Taylor expansion at $\varepsilon = 0$ yields

$$(15) \quad t_- + s_i = \frac{(b - a)(2a + 1)}{2} \varepsilon^3 + O(\varepsilon^5), \quad s_i - t_- = 2\varepsilon + O(\varepsilon^3),$$

$$(16) \quad s_i - t_+ = \frac{(b - a)(2a - 1)}{2} \varepsilon^3 + O(\varepsilon^5), \quad t_+ + s_i = 2\varepsilon + O(\varepsilon^3).$$

Here $a = 3i \geq 3$, so all four displayed quantities are positive for small ε . Together with (13), this proves that $\mathcal{B}_i$ meets both legs of N_k properly.

Finally, by (6), every narrow crossbar supporting line passes through

$$X_\varepsilon = (\cos \varepsilon, 0).$$

Its coordinate on $\mathcal{B}_i$ is

$$n_i \cdot (X_\varepsilon - M_i) = \sin(a\varepsilon)(1 - \cos \varepsilon) = O(\varepsilon^3),$$

whereas $s_i = \varepsilon + O(\varepsilon^3)$. Hence X_ε lies strictly inside both $\mathcal{B}_i$ and $\mathcal{B}_k$ for small ε . We have obtained the seven entries of M_{NN} . $\square$

Remark 5.2. The unperturbed narrow crossbars are concurrent at X_ε . This is a convenient feature of the proof, not a claimed general-position configuration. The concurrence will be removed in section 6 without changing any of the strict incidences above.

5.2. Narrow–wide pairs.

Lemma 5.3. *For $1 \leq i \leq p$ and $1 \leq j \leq q$, and all sufficiently small $\varepsilon > 0$, the component incidence matrix of (N_i, W_j) contains M_{NW} from (4). Thus the pair has eight proper intersections.*

Proof. Set $a = a_i$, $L = L_j$, $D = D_j$, and abbreviate

$$(17) \quad \mu = \tan \theta_j = \frac{2q_j}{1 - q_j^2}, \quad \nu = \tan \frac{\theta_j}{2} = q_j.$$

We first meet the wide low leg. The determinant of its direction with the σ -leg of N_i is

$$\sin((L - a - \sigma)\varepsilon),$$

and

$$L - a - \sigma \geq (4m + 1) - (3m + 1) = m > 0.$$

If r_σ and s_σ are the two ray parameters at their intersection, the determinant formula for two lines gives

$$(18) \quad \varepsilon r_\sigma \longrightarrow \frac{1}{L - a - \sigma}, \quad \varepsilon s_\sigma \longrightarrow \frac{1}{L - a - \sigma}.$$

Both are positive, so each narrow leg meets the wide low leg properly.

The remaining six required incidences are visible in the blow-up at C ,

$$(X, Y) = \left(\frac{x - 1}{\varepsilon}, \frac{y}{\varepsilon} \right).$$

The two narrow legs tend to $Y = -1$ and $Y = 1$, while $\mathcal{B}_i$ tends to the open vertical segment $X = 0$, $-1 < Y < 1$. The high leg and crossbar of W_j tend to

$$(19) \quad \mathcal{H}_j : Y - h = -\mu(X + D),$$

$$(20) \quad \mathcal{C}_j : Y - h = -\nu(X + D), \quad X > -D,$$

respectively. The inequality in (20) is the limiting interior direction from the high endpoint toward the low endpoint.

For a narrow leg $Y = \sigma$, its intersection with (20) satisfies

$$X + D = \frac{h - \sigma}{\nu} > 0$$

because $h = 3/2 > 1$. Hence both narrow legs meet the wide bar in its interior. In this chart the apex of the high ray escapes to infinity along $\mathcal{H}_j$ in the direction opposite the ray. Consequently every fixed finite point of $\mathcal{H}_j$, including its intersections with the two narrow legs, lies in the ray's interior for small ε .

At $X = 0$, the limiting heights of the narrow-bar intersections are

$$(21) \quad h - D\mu = h - \frac{2}{1 - q_j^2}, \quad h - D\nu = h - 1 = \frac{1}{2}.$$

Since $0 < q_j \leq 1/4$,

$$-\frac{19}{30} = \frac{3}{2} - \frac{32}{15} \leq h - D\mu < -\frac{1}{2},$$

so both values in (21) lie strictly between -1 and 1 . The second intersection also has $X + D = D > 0$, placing it inside the wide bar. By lemma 3.1, the narrow bar meets both the wide high leg and the wide bar properly. Together with (18), these are exactly the eight required entries of M_{NW} . $\square$

5.3. Wide-wide pairs.

Lemma 5.4. *For $1 \leq j < k \leq q$ and all sufficiently small $\varepsilon > 0$, the component incidence matrix of (W_j, W_k) contains M_{WW} from (4). In particular, the pair has at least seven proper intersections.*

Proof. Write

$$\mu_r = \tan \theta_r = \frac{2q_r}{1 - q_r^2}, \quad \nu_r = q_r \quad (r = j, k).$$

Because $q_k \leq q_j/4$ and $q_j \leq 1/4$,

$$(22) \quad \mu_j > \nu_j > \mu_k > \nu_k > 0.$$

Indeed, only the middle inequality needs comment, and

$$\mu_k \leq \frac{q_j/2}{1 - q_j^2/16} < q_j = \nu_j.$$

In the blow-up at C , each high leg or crossbar of W_r has a limiting line through $(-D_r, h)$ of slope $-s$, where s is respectively μ_r or ν_r . Consider one such line for $r = j$ with slope $-s$ and one for $r = k$ with slope $-t$. By (22), $s > t$. If their intersection has displacements

$$U_j = X + D_j, \quad U_k = X + D_k,$$

then, with $\Delta = D_k - D_j > 0$,

$$(23) \quad U_j = \frac{t\Delta}{s - t} > 0, \quad U_k = \frac{s\Delta}{s - t} > 0.$$

Whenever one of the two components is a crossbar, the corresponding positive displacement places the intersection strictly beyond its high endpoint toward its low endpoint. The displacement is finite in the blow-up, whereas the other endpoint escapes to infinity, so it is strictly inside the bar. The apex of a high leg escapes to infinity along its limiting line in the direction opposite the ray, so every fixed finite point on that line is in the ray's interior for small ε . Thus (23) proves the four high/bar incidences

$$(\text{high}, \text{high}), \quad (\text{high}, \text{bar}), \quad (\text{bar}, \text{high}), \quad (\text{bar}, \text{bar}).$$

It remains to show that the low leg of W_j meets all three components of W_k . Use the second blow-up

$$(x, Y) = \left(x, \frac{y + 1}{\varepsilon}\right).$$

By (10)–(11), the low leg of W_r tends to

$$(24) \quad Y = L_r(x - A_r), \quad x > A_r.$$

The portion of the high ray of W_k at height $O(\varepsilon)$ tends to the vertical half-line

$$x = A_k, \quad Y > 0.$$

The low endpoint of the k th crossbar has the more precise limit

$$(25) \quad (B_k, \kappa_k), \quad \kappa_k = \frac{L_k}{\sin \theta_k} = \frac{L_k(D_k + q_k)}{2},$$

because $((K_k)_y + 1)/\varepsilon = R_k \sin \lambda_k/\varepsilon \rightarrow L_k/\sin \theta_k$. The portion of the crossbar near that endpoint tends to the vertical half-line $x = B_k$, $Y > \kappa_k$.

Since $j < k$, we have $A_j < A_k < B_k$ and $L_j < L_k$. The limiting intersections of the j th low leg with the k th high leg and crossbar are respectively

$$(26) \quad (A_k, L_j(A_k - A_j)), \quad (B_k, L_j(B_k - A_j)).$$

The first has positive height and is forward of the j th apex. For the second, note that $k \geq 2$, $D_j \leq D_k/4$, $q_k^2 \leq 1/256$, and

$$\frac{L_k}{L_j} = 1 + \frac{k-j}{4m+j} < \frac{5}{4}.$$

Therefore

$$\begin{aligned} B_k - A_j &= D_k - \frac{D_j}{2} + \frac{q_j}{2} \geq \frac{7D_k}{8}, \\ \frac{\kappa_k}{L_j} &= \frac{L_k}{L_j} \frac{D_k + q_k}{2} < \frac{5}{4} \cdot \frac{257}{256} \cdot \frac{D_k}{2} < \frac{7D_k}{8}. \end{aligned}$$

Thus $L_j(B_k - A_j) > \kappa_k$, so the second point in (26) lies strictly inside the limiting crossbar. Both incidences are therefore proper for small ε .

Finally, the two limiting low lines in (24) meet at

$$(27) \quad x_* = \frac{L_k A_k - L_j A_j}{L_k - L_j} = A_k + \frac{L_j(A_k - A_j)}{L_k - L_j} > A_k > A_j.$$

Thus the low-low intersection lies forward on both rays. These three low incidences and the four incidences from (23) give M_{WW} . $\square$

6. GENERIC RATIONAL PERTURBATION AND COMPLETION OF THE PROOF

The preceding construction was chosen for transparent limiting geometry, not for general position. In particular, the narrow crossbars concur as noted in remark 5.2. We now remove every degeneracy at once.

Lemma 6.1 (Generic rational realization). *For every $p, q \geq 1$, there is a generic arrangement of $p + q$ constrained A 's, with rational apices, rational positive radii, and rational unit leg directions, such that every narrow-wide pair has at least eight proper intersections and every same-family pair has at least seven.*

Proof. Choose $\varepsilon > 0$ small enough that all strict component-domain inequalities proved in lemmas 5.1, 5.3 and 5.4 hold simultaneously. They define an open neighborhood $\mathcal{U}$ of the resulting point in the parameter space of apices, radii, and leg directions.

Work in a local rational chart for each unit direction,

$$t \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$

After denominators are cleared, each unwanted event—supporting-line parallelism, a foreign component through an apex or crossbar endpoint, or the coincidence of two distinctly labelled finite crossings—is contained in the zero set of a polynomial in the apex coordinates, radii, and chart parameters. Only finitely many labels occur. None of these polynomials is identically zero on $\mathcal{U}$: an arbitrarily small independent change of one participating apex, radius, or direction destroys the corresponding event. The bad locus is therefore a finite union of proper algebraic zero sets and has empty interior.

Rational points are dense in the chart. Since the bad locus is closed in $\mathcal{U}$ and cannot contain a dense subset of $\mathcal{U}$, there is a rational point of $\mathcal{U}$ outside it. It gives a generic arrangement with

rational apices, positive rational radii, and rational unit directions. All required incidences persist because their inequalities are strict. $\square$

Proof of theorem 1.1. The cases $n = 0, 1$ give 1 and 3 regions directly. Let $n \geq 2$ and take

$$p = \left\lfloor \frac{n}{2} \right\rfloor, \quad q = \left\lceil \frac{n}{2} \right\rceil.$$

Apply lemma 6.1. The pq cross-family pairs contribute at least eight crossings each, while the remaining pairs contribute at least seven. Because the arrangement is generic, all these crossings are globally distinct, and therefore

$$\begin{aligned} I &\geq 8pq + 7 \left(\binom{p}{2} + \binom{q}{2} \right) \\ &= 7 \binom{n}{2} + pq = 7 \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

The reverse inequality is theorem 2.3. Hence equality holds, and proposition 2.1 gives (1). $\square$

Corollary 6.2. *For every $p, q \geq 1$, there is a generic rational arrangement of $p + q$ constrained A's in which every cross-family pair has exactly eight crossings and every same-family pair has exactly seven. It has*

$$7 \binom{p+q}{2} + pq + 2(p+q) + 1$$

regions.

Proof. The cross-family pairs have exactly eight crossings by proposition 2.2 and lemma 6.1. If a same-family pair also had eight, then that pair together with any object from the other family would be three pairwise eight-crossing A's, contrary to proposition 2.2. The region count follows from proposition 2.1. $\square$

Corollary 6.3. *The sequence begins*

$$1, 3, 13, 30, 55, 87, 127, 174, 229, 291, 361, 438, 523, 615, 715, 822, 937, \dots$$

For equality arrangements, the graph of eight-crossing pairs is necessarily $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ and every within-part pair has exactly seven crossings.

7. EXACT COMPUTATIONAL CERTIFICATES

The proof above does not depend on computation. During discovery, however, exact generic rational constructions were obtained separately for every $4 \leq n \leq 16$. They provide useful independent checks of the equality pattern. The supplementary archive contains the certificates listed in table 1 and a single verifier using only Python's `fractions.Fraction` class.

For every certificate the verifier checks:

- (i) equal positive squared norms of the two leg directions and a positive radius for every A;
- (ii) all nine component-pair predicates for every pair of A's, with strict ray and crossbar domains;
- (iii) the balanced seven/eight pair-crossing matrix;
- (iv) absence of cross-shape supporting-line parallelism;
- (v) absence of every foreign apex or crossbar-endpoint incidence; and
- (vi) global distinctness of all proper rational crossings.

TABLE 1. Exact rational certificates included in the supplement.

n	part sizes	crossings	regions	file
4	2 + 2	46	55	certificates/n04.json
5	2 + 3	76	87	certificates/n05.json
6	3 + 3	114	127	certificates/n06.json
7	3 + 4	159	174	certificates/n07.json
8	4 + 4	212	229	certificates/n08.json
9	4 + 5	272	291	certificates/n09.json
10	5 + 5	340	361	certificates/n10.json
11	5 + 6	415	438	certificates/n11.json
12	6 + 6	498	523	certificates/n12.json
13	6 + 7	588	615	certificates/n13.json
14	7 + 7	686	715	certificates/n14.json
15	7 + 8	791	822	certificates/n15.json
16	8 + 8	904	937	certificates/n16.json

A second standard-library script implements a fully rational, nongeneric version of the two-family base construction. It checks the exact masks

$$110110111, \quad 111111011, \quad 111011011$$

for narrow–narrow, narrow–wide, and wide–wide pairs respectively. The archived run covers every balanced size $2 \leq n \leq 64$. This is a regression check rather than a logical input to the all- n proof.

8. CONCLUDING REMARKS

The upper and lower arguments isolate two different kinds of structure. The upper bound remembers only whether a pair has seven or eight crossings; its equality case is forced by Mantel’s theorem. The lower bound is more geometric: the two color classes require different scales and different local models. The narrow family supplies four leg–leg crossings and three bar incidences at a common macroscopic location. The wide nested fan uses an exponentially separated opening parameter $q_j = 4^{-j}$ and a linearly ordered drift $L_j \varepsilon$. The scale separation makes the three pair types independent enough to verify by elementary limits.

More generally, corollary 6.2 realizes every complete bipartite graph $K_{p,q}$ as an eight-crossing graph, with seven crossings on every remaining pair. Thus the three masks in (4) are compatible at arbitrary scale. The two blow-up centers separate the relevant incidence constraints into two elementary line fans; the generic perturbation then converts their strict local inequalities into a global arrangement.

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RESEARCH FELLOW, LOSSFUNK

Email address: `siddharthamahajan03@gmail.com`

LOSSFUNK

Email address: `paras@lossfunk.com`